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THE SPACE OF RETRACTIONS OF THE 2-SPHERE AND THE ANNULUS

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Abstract. Given a manifold M, there is an embedding Λ of M into the space of retractions of M, taking each point to the retraction of M to that point. Considering Λ as a map into the connected component containing its image, we prove that Λ is a weak homotopy equivalence for two choices of M, namely, the 2-sphere and the annulus.

Introduction. In [3], Borsuk introduced the study of spaces of retractions and proved a theorem, which, when specialized to 2-manifolds, says that the space of retractions of a closed disk is contractible (in itself).

Given a manifold M, there is a natural isometric embedding Λ of M into the space of retractions of M, defined by setting $\Lambda(x)$ equal to the retraction taking M to x, for any $x \in M$. Let $\mathscr{L}(M)$ denote the component containing the image of Λ. Then $ev \circ \Lambda$ is the identity map on M, where ev maps any retraction to the retraction evaluated at some basepoint of M. Hence the induced homomorphism $\Lambda_*: \pi_n(M) \to \pi_n(\mathscr{L}(M))$ is injective, for all n. In §2 and §3, we prove that Λ_* is surjective for two choices of M, namely, the 2-sphere and the 2-dimensional annulus. Thus we prove that Λ in these two cases is a weak homotopy equivalence. We also show that for this pair of 2-manifolds, $\mathscr{L}(M)$ is just the space of non-deformation retractions of M. Hence these results show that the space of non-deformation retractions of the 2-sphere (resp. the annulus) has the same homotopy groups as the 2-sphere (resp. the annulus).

The proofs rely heavily on a selection theorem of E. Michael, which is stated in §1. The proof in §3 requires certain results from conformal mapping theory which are also stated in §1.

1. **Preliminaries.** Throughout this paper we use certain standard notation, namely, E^n for Euclidean n-space, S^n for the unit n-sphere in E^{n+1}, B^n for the unit

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n-ball in E^n, and I for the closed unit interval [0, 1]. In addition, we use C_α to denote the circle in E^2 with center at the origin and radius α, and A^2 to denote the closed annulus bounded by C_1 and C_2. The symbol $A(J, K)$ denotes the closed annular region bounded by simple closed curves J and K, where J lies in the bounded complementary domain of K. Thus $A^2 = A(C_1, C_2)$.

A *retraction* of a space X is an idempotent map φ of X to itself, and the space $\varphi(X)$, denoted im (φ), is called a *retract* of X. All function spaces in this paper will be given the compact-open topology, and since we treat only compact metric spaces, this topology will be the same as that given by the sup-metric. In addition to the notation $\mathscr{L}(M)$ defined in the introduction, we use $\mathscr{R}(X)$, $\mathscr{D}(X)$, and $\mathscr{N}(X)$, respectively, for the space of retractions, deformation retractions, and non-deformation retractions of X, respectively. A map $f\colon X \to Y$ is a *weak homotopy equivalence* if the induced maps $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ are isomorphisms for all n, where x_0 is some basepoint of X. The existence of a weak homotopy equivalence implies that the two spaces involved have the same homotopy and integral singular homology groups [16, p. 406].

To motivate later work, we now present a modified proof of a theorem of Borsuk [3, p. 197]. The proof is almost a direct translation of the proof by Alexander [1] that the space of homeomorphisms of an n-ball, keeping the boundary fixed, is contractible. Since we are only concerned with 2-manifolds, we give the theorem for the disk, even though it is true for much more general spaces.

1.1. THEOREM (BORSUK). *The space $\mathscr{R}(B^2)$ is contractible in itself.*

Proof. For each $t \in I$, define a retraction ρ_t of B^2 by letting ρ_t be the identity on the disk inside C_{1-t}, and letting it map $A(C_{1-t}, C_1)$ radially to C_{1-t}. Next, for $0 \leqq t < 1$, define a homeomorphism $h_t\colon B^2 \to \rho_t(B^2)$ by $h_t(u) = (1-t)u$, for $u \in B^2$. Finally, define a contraction Θ_t of $\mathscr{R}(B^2)$ by

$$\Theta_t(\varphi) = h_t \circ \varphi \circ h_t^{-1} \circ \rho_t, \quad \text{for } 0 \leqq t < 1,$$

and $\Theta_1(\varphi) = \rho_1$. We see that $\Theta_t(\varphi)$ always belongs to $\mathscr{R}(B^2)$, since $h_t \circ \varphi \circ h_t^{-1}$ is an idempotent map whose image is contained in the image of ρ_t. Only continuity for $t = 1$ needs verifying, and this presents no problems. Q.E.D.

As a generalization of the technique in the previous proof, we have the following:

1.2. REMARK (ELBOWROOM CONSTRUCTION). Let M be a manifold with boundary ∂M and let $\partial M \times [0, 2]$ be a collar of the boundary. For $t \in I$, let ρ_t be the retraction of M which is the identity outside $\partial M \times [0, t]$, and projects $\partial M \times [0, t]$ to $\partial M \times \{t\}$. Let $h_t\colon M \to \rho_t(M)$ be the homeomorphism given by the identity outside $\partial M \times [0, 2]$ and by mapping $\partial M \times [0, 2]$ linearly to $\partial M \times [t, 2]$. Then the homotopy Θ_t given by $\Theta_t(\varphi) = h_t \circ \varphi \circ h_t^{-1} \circ \rho_t$ provides a deformation of $\mathscr{R}(M)$ in itself such that the image of Θ_1 consists of retractions of M whose images do not meet ∂M.

For future reference, we now quote a selection theorem of E. Michael [14, p. 563]. The theorem is stated here in a weak form, as in [11, p. 528].

1.3. THEOREM (MICHAEL). *Suppose that B and X are metric spaces, B is (metrically) topologically complete, the (covering) dimension of X does not exceed n, and that Y is a closed subspace of X. Suppose, further, that τ is an open mapping of B onto X such that the collection of (point) inverses under τ is equi-LC^{n-1} (see Definition* 1.4), *and that we have a partial mapping e of Y into B such that, for $y \in Y$, $e(y) \in \tau^{-1}(y)$. Then there is a neighborhood U of Y such that e may be extended to a mapping e^* of U into B such that, for $x \in U$, $e^*(x) \in \tau^{-1}(x)$. If each inverse under τ has vanishing homotopy groups of order $\leqq n-1$, then U may be taken to be the entire space X.*

1.4. DEFINITION. In a metric space B, let $N(w, \varepsilon)$ denote the open neighborhood with center at $w \in B$ and radius ε. Recall that B is LC^n if for $w \in B$ and $\varepsilon > 0$, there is a $\delta > 0$ such that each mapping of an m-sphere ($m \leqq n$) into $N(w, \delta)$ is nullhomotopic in $N(w, \varepsilon)$. Using the notation of Theorem 1.3, the collection of inverses under τ is equi-LC^n if, for $x \in X$, $w \in \tau^{-1}(x)$, and $\varepsilon > 0$, there is a $\delta > 0$ such that, if $x' \in X$, then every mapping of an m-sphere ($m \leqq n$) into $\tau^{-1}(x') \cap N(w, \delta)$ is nullhomotopic in $\tau^{-1}(x') \cap N(w, \varepsilon)$.

We wish to state some results from conformal mapping theory which will be needed later. In the rest of this section, whenever the index i occurs, we assume that it ranges over the set of positive integers. In what follows, we have omitted the word "respectively" a number of times.

Let G and G_i be open, connected, simply connected subsets of E^2 with boundaries the simple closed curves J and J_i. Suppose these curves are images of similarly oriented homeomorphisms f and f_i with domain C_1. Let p be a point of G and p_i a point of G_i. Then there are homeomorphisms h and h_i of B^2 onto $G \cup J$ and $G_i \cup J_i$ which are conformal on the interior of B^2 and uniquely determined by requiring that they map the origin to p and p_i and that their derivatives be real and positive at the origin [13, p. 70]. Moreover, if $\{p_i\}$ converges to p and $\{f_i\}$ converges uniformly to f on C_1, then $\{h_i\}$ converges uniformly to h on B^2. (See [4], [5], [6, p. 191], and [9, p. 27]. We shall sometimes refer to this last fact as the "continuity property.") Finally, there are unique (not necessarily conformal) homeomorphisms F and F_i of B^2 onto $G \cup J$ and $G_i \cup J_i$ such that F extends f, F_i extends f_i, F and F_i map the origin to p and p_i, and the sequence $\{F_i\}$ converges uniformly to F on B^2.

Similar results hold for annular regions. Let f, f_i, g, and g_i be similarly oriented homeomorphisms onto simple closed curves J, J_i, K, and K_i, where f and f_i have domain C_1, g and g_i have domain C_2, and J and J_i lie in the bounded complementary domains of K and K_i. First, there is, for some $r > 1$, a homeomorphism of the annulus $A(C_1, C_r)$ onto $A(J, K)$, which is conformal on the interior and is uniquely determined by the orientation of the boundary and the image of one boundary point [6, p. 38]. Next, there are homeomorphisms F and F_i which map A^2 onto $A(J, K)$ and $A(J_i, K_i)$, which extend the maps f, f_i, g, and g_i, and which are uniquely determined by the angle change along the image of the closed line segment

I' from (1, 0) to (2, 0). (This means the net angle change with respect to a point inside J. For a careful definition, see [11, p. 522].) Finally, we have the continuity property: if $\{f_i\}$ converges uniformly to f on C_1, $\{g_i\}$ converges uniformly to g on C_2, and the angle change along the arcs $F_i(I')$ converges to the angle change along $F(I')$, then the sequence $\{F_i\}$ converges uniformly to F on A^2. (See [6], [11], and [15, p. 45].)

2. **The 2-sphere.** In this section, the goal is to prove the following theorem.

2.1. THEOREM. *The embedding* $\Lambda: S^2 \to \mathscr{L}(S^2)$ *is a weak homotopy equivalence. In addition,* $\mathscr{L}(S^2) = \mathscr{N}(S^2)$.

It will follow from work later in this section that $\mathscr{N}(S^2)$, the space of non-deformation retractions, is pathwise connected. It is then easy to see that $\mathscr{L}(S^2) = \mathscr{N}(S^2)$, the identity map being the only deformation retraction. As stated in the introduction, we need only show that the induced map

$$\Lambda_*: \pi_n(S^2, u_0) \to \pi_n(\mathscr{N}(S^2), \Lambda(u_0))$$

is surjective, where u_0 is some basepoint of S^2.

Given a retraction φ in $\mathscr{N}(S^2)$, we shall show in this section how to construct a canonical deformation in $\mathscr{N}(S^2)$ from φ to $\Lambda(a(u))$, where a is the antipodal map on S^2 and u is a point of $S^2 \backslash \operatorname{im}(\varphi)$. If we could just continuously select, for each $\varphi \in \mathscr{N}(S^2)$, a point from $S^2 \backslash \operatorname{im}(\varphi)$, then we could easily show that the map Λ above is a homotopy equivalence, with homotopy inverse the evaluation map taking φ to $\varphi(u_0)$. In this section, we use Michael's Theorem to select such points from $S^2 \backslash \operatorname{im}(\varphi)$, but a finite-dimensionality condition in the theorem limits our result to proving that Λ is a weak homotopy equivalence.

In order to prove Λ_* surjective, let $\Phi: (I^n, \partial I^n) \to (\mathscr{N}(S^2), \Lambda(u_0))$ be a map, where I^n is the unit n-cube and ∂I^n is its boundary. For each $x \in I^n$, set $B(x) = S^2 \backslash \operatorname{im}(\Phi(x))$, and let $B \subset I^n \times S^2$ be the set of all points (x, u), where $x \in I^n$ and $u \in B(x)$.

2.2. LEMMA. *The set* B *defined above is an open subspace of* $I^n \times S^2$.

Proof. Let $(x_0, w_0) \in B$, i.e., let $w_0 \in B(x_0)$. Choose a number $\eta > 0$ so that $\operatorname{dist}(w_0, \operatorname{im}(\Phi(x_0))) \geqq \eta$. Choose $\delta > 0$ such that if $x \in I^n$ and $d(x, x_0) < \delta$, then $d(\Phi(x), \Phi(x_0)) < \eta/2$. (Note that the first "$d$" represents the usual distance in E^n, while the second "d" is being used for the sup-metric on $\mathscr{N}(S^2)$.) Define an open neighborhood U of (x_0, w_0) in $I^n \times S^2$ by letting U equal the set of all (x, w) such that $d(x, x_0) < \delta$ and $d(w, w_0) < \eta/2$. In order to show that $U \subset B$, we choose $(x, w) \in U$ and show that $(x, w) \in B$ or $w \in B(x)$. It will clearly suffice to show $\operatorname{dist}(w, \operatorname{im}(\Phi(x))) > 0$. Let $q \in \operatorname{im}(\Phi(x))$ and note that $\Phi(x)(q) = q$. Thus we have

$$d(w_0, q) \geqq d(w_0, \Phi(x_0)(q)) - d(\Phi(x_0)(q), \Phi(x)(q)) > \eta - (\eta/2) = \eta/2.$$

Hence

$$d(w, q) \geqq d(w_0, q) - d(w_0, w) > \eta/2 - \eta/2 = 0. \qquad \text{Q.E.D.}$$

Now consider the projection τ_1 of $I^n \times S^2$ onto I^n, and let τ be τ_1 restricted to B, so that $\tau^{-1}(x) = \{x\} \times B(x)$.

2.3. LEMMA. *The spaces and map defined above satisfy the hypotheses of Theorem* 1.3, *i.e.*,

(a) *the space B is topologically complete,*

(b) *the map τ is an open surjection,*

(c) *for each $x \in I^n$, the space $\tau^{-1}(x)$ has vanishing homotopy groups of order $\leqq n-1$, and*

(d) *the collection of inverses under τ, i.e., the collection of sets $\{x\} \times B(x)$ for $x \in I^n$, is equi-LC^{n-1}. (See Definition* 1.4.)

Proof. Using Lemma 2.2, the fact that τ_1 is open, and the fact that im $(\Phi(x))$ is an absolute retract, everything is clear except possibly (d). To prove (d), note first that equi-LC^n is a topological property. Thus choose a metric on $I^n \times S^2$ which yields open neighborhoods of the form: an open "box" in I^n crossed with an open disk in S^2. The equi-LC^n property then becomes a triviality. Q.E.D.

Proof of Theorem 2.1. We are in a position to apply Theorem 1.3, with X in the theorem equal to I^n, Y equal to ∂I^n, B and τ of the theorem as above, and $e: \partial I^n \to B$ defined by $e(y) = (y, a(u_0))$, for each $y \in \partial I^n$, where a is the antipodal map on S^2 and u_0 is the basepoint of S^2. Note that $(y, a(u_0)) \in B$ because Φ maps ∂I^n to $\Lambda(u_0)$. Theorem 1.3 gives an extension e^* of e, mapping I^n into B. If τ_2 is the projection of $I^n \times S^2$ onto S^2, then the map $\tau_2 \circ e^*: I^n \to S^2$ satisfies $\tau_2 \circ e^*(x) \in B(x)$. Thus, corresponding to each $x \in I^n$, $\tau_2 \circ e^*$ selects a point from the complement of im $(\Phi(x))$.

We now construct a (basepoint-preserving) homotopy in $\mathcal{N}(S^2)$ from the map Φ to the map $\Lambda \circ a \circ \tau_2 \circ e^*$. Since the latter map is a representative of the image of $a \circ \tau_2 \circ e^*$ under Λ_*, this will show that Λ_* is surjective. A similar construction can be used to show that $\mathcal{N}(S^2)$ is pathwise connected.

For notational convenience, let x' stand for the point $\tau_2 \circ e^*(x)$ in $S^2 \backslash$im $(\Phi(x))$, where $x \in I^n$. A compactness argument shows that there is a number $\beta > 0$ such that the great circle distance between x' and im $(\Phi(x))$ is greater than β for all $x \in I^n$.

In S^2, let $C(x)$ stand for the circle with center x' and radius β (great circle distance). The circle $C(x)$, along with the component of $S^2 \backslash C(x)$ containing x', is disjoint from im $(\Phi(x))$. Define a collection of homeomorphisms $h_t(x)$ of S^2 onto itself, for each $0 \leqq t < 1$ and each $x \in I^n$. Let $h_t(x)$ take $C(x)$ to the circle with center x' and radius $(1-t)\beta + t\pi$, let $h_t(x)$ fix x' and its antipode $a(x')$, and extend $h_t(x)$ linearly along great circles through x'. The map $h_0(x)$ is the identity on S^2, and the effect of $h_t(x)$, as t tends to 1, is to take the component of $S^2 \backslash C(x)$ not containing x' to the point $a(x')$. It is clear that these homeomorphisms are jointly continuous in the sup-metric in both variables t and x.

We can now define a homotopy Θ_t in $\mathscr{N}(S^2)$ by setting, for each $x \in I^n$,

$$\Theta_t(x) = h_t(x) \circ \Phi(x) \circ h_t(x)^{-1}, \quad \text{for } 0 \leqq t < 1,$$

and setting $\Theta_1(x) = \Lambda(a(x')) = \Lambda \circ a \circ \tau_2 \circ e^*(x)$. Here $\Theta_0(x) = \Phi(x)$, and $\Theta_t(y) = \Lambda(u_0)$, for $y \in \partial I^n$ and $t \in I$. Also $\Theta_t(x) \in \mathscr{N}(S^2)$ for any t and x, and Θ_t is clearly continuous when $t \neq 1$.

The continuity of Θ_t when $t = 1$ follows from the special nature of the construction. If $D(x)$ denotes the component of $S^2 \backslash C(x)$ containing $a(x')$, then im $(\Phi(x))$ is contained in $D(x)$. Thus im $(\Theta_t(x))$ is contained in $h_t(x)(D(x))$, and the latter set is an open disk centered at $a(x')$ whose radius we can make as small as we like by taking t sufficiently close to 1. Q.E.D.

REMARK. The methods in this section cannot be used for S^m $(m > 2)$ because the complement of a retract of S^m need not have vanishing homotopy groups. In fact, the solid Alexander Horned Sphere is an absolute retract, and so is a retract of S^3, but its complement is not simply connected.

3. **The annulus.** Recall that the annulus A^2 is the set $A(C_1, C_2)$ consisting of points $u \in E^2$ such that $1 \leqq |u| \leqq 2$. In this section we use methods very similar to those in §2 to obtain the following result.

3.1. THEOREM. *The embedding $\Lambda\colon A^2 \to \mathscr{L}(A^2)$ is a weak homotopy equivalence. In addition, $\mathscr{L}(A^2) = \mathscr{N}(A^2)$.*

If φ is a retraction of A^2, then $\varphi^2 = \varphi$, so that the degree of φ is either 1 or 0, and hence φ is either homotopic to the identity or to a constant map. The first Čech cohomology group of the image of φ is thus determined, depending on whether φ is a deformation retraction or not, and correspondingly the image of φ separates E^2 into two components, or its complement is connected. In any compact manifold, there is a number ε such that any two self maps whose distance apart is less than ε are homotopic. Thus $\mathscr{N}(A^2)$ and $\mathscr{D}(A^2)$ are at least a distance ε apart. (In fact, it is not hard to show that they are a distance 2 apart.) Finally, it will follow from work later in this section that $\mathscr{N}(A^2)$ is pathwise connected, so that $\mathscr{L}(A^2) = \mathscr{N}(A^2)$.

Using Remark 1.2 and some conformal mapping theory, one can also prove that $\mathscr{D}(A^2)$ is pathwise connected. A detailed study of the space of deformation retractions of any compact 2-manifold will appear elsewhere.

As in §2, we can finish the proof of Theorem 3.1 by showing that the induced map Λ_* is surjective. The idea of the proof is similar to that of Theorem 2.1, but here most of the work will involve verifying that the hypotheses of Theorem 1.3 are satisfied.

Let $\Phi\colon (I^n, \partial I^n) \to (\mathscr{N}(A^2), \Lambda(3/2, 0))$ be a map, where $(3/2, 0)$ is the basepoint of A^2. Using the elbowroom construction of Remark 1.2, we can assume that im (Φ) consists of retractions whose images do not meet ∂A^2. It turns out that the crucial problem in this section is to produce continuously, for each $x \in I^n$, an arc in $A^2\backslash\mathrm{im}\,(\Phi(x))$ from C_1 to C_2, lying except for its endpoints in the interior of A^2. We can then produce a homotopy from Φ to a map in the image of Λ_*, proving that Λ_* is surjective.

For any $\varphi \in \mathscr{N}(A^2)$, denote $A^2\backslash\mathrm{im}\,(\varphi)$ by $M(\varphi)$. Let $\mathscr{H}(I, A^2)$ stand for the space of homeomorphisms $h\colon I \to A^2$ such that h takes 0 to C_1, 1 to C_2, and the interior of I to the interior of A^2. Similarly, let $\mathscr{H}(I, M(\varphi))$ be the space of homeomorphisms $h\colon I \to M(\varphi)$ satisfying the other three conditions. Define $\mathscr{B}$ to be the subset of $I^n \times \mathscr{H}(I, A^2)$ consisting of points (x, h) such that $x \in I^n$ and $h \in \mathscr{H}(I, M(\Phi(x)))$. Finally, let τ_1 be the projection of $I^n \times \mathscr{H}(I, A^2)$ onto I^n, and let τ be the restriction of τ_1 to $\mathscr{B}$, so that $\tau^{-1}(x) = \{x\} \times \mathscr{H}(I, M(\Phi(x)))$.

3.2. Theorem. *The spaces and map defined above satisfy the hypotheses of Theorem* 1.3, *i.e.*,

(a) *the space $\mathscr{B}$ is topologically complete*,

(b) *the map τ is an open surjection*,

(c) *for each $x \in I^n$, the space $\tau^{-1}(x)$ has vanishing homotopy groups of order $\leqq n-1$, and*

(d) *the collection of inverses under τ is equi-LC^{n-1}.*

The theorem will follow from the next two lemmas without much difficulty.

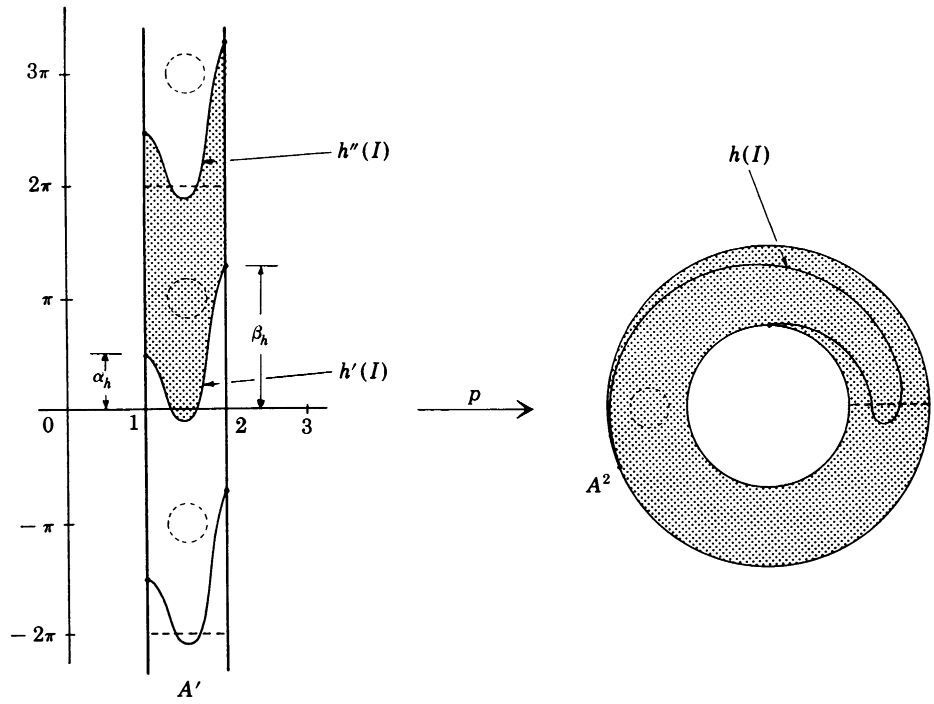
3.3. Lemma. *For each $\varphi \in \mathscr{N}(A^2)$, the space $\mathscr{H}(I, M(\varphi))$ is contractible in itself.*

Proof. Unless the contrary is explicitly stated, coordinates of all points will be rectangular.

Since $\mathrm{im}\,(\varphi)$ is contractible, for $\varphi \in \mathscr{N}(A^2)$, it is easy to see that $M(\varphi)$ is homeomorphic to A^2 minus the closed disk with radius 1/4 and center $(-3/2, 0)$. Denote this set by Z. Let $\mathscr{H}(I, Z)$ be the subspace of $\mathscr{H}(I, A^2)$ consisting of those homeomorphisms which map into Z. We shall show that $\mathscr{H}(I, Z)$ is contractible.

Let $A' = \{(u_1, u_2) \in E^2 : 1 \leqq u_1 \leqq 2\}$ be the universal covering space of A^2, where the covering projection p is defined by $p(u_1, u_2) = (u_1 \cos u_2, u_1 \sin u_2)$. For any $h \in \mathscr{H}(I, Z)$, the interior of A^2 minus the image of h is an open disk in A^2. Since it is simply connected, it is covered in A' by disjoint copies which project homeomorphically into A^2. In A' choose the copy of this open disk which contains the closed disk with radius 1/4 and center $(3/2, \pi)$. Lift h to maps h' and h'' of I into A' whose images lie on the boundary of the chosen open disk, such that the second coordinate of $h'(t)$ is 2π smaller than the second coordinate of $h''(t)$, for $t \in I$. (Please refer to the Figure.) Let α_h be the second coordinate of $h'(0)$ and let β_h be the second coordinate of $h'(1)$.

We now define, for each $h \in \mathscr{H}(I, Z)$, an isotopy $H_t(h)$ of A'. The first coordinates of all points are left fixed, and $H_t(h)$ fixes points of A' with first coordinate u_1 between 9/8 and 15/8. Points with first coordinate $u_1 = 1$ have their second coordinate decreased by $t\alpha_h$, and for u_1 between 1 and 9/8, the amount of decrease tends linearly to zero when $u_1 = 9/8$. Similarly, points with $u_1 = 2$ have their second



coordinate decreased by $t\beta_h$, and for u_1 between 15/8 and 2, the amount of decrease tends linearly to zero at $u_1 = 15/8$. The effect of $H_t(h)$, as t goes from 0 to 1, is to continuously shift $h'(0)$ and $h'(1)$ to the points (1, 0) and (2, 0), respectively. Let $\mathscr{H}'(I, Z)$ be the subspace of $\mathscr{H}(I, Z)$ consisting of maps h such that $h'(0) = (1, 0)$ and $h'(1) = (2, 0)$. (Notice that $\mathscr{H}'$ is defined using the special "lifted" map h'.) There are various ways of seeing that any $h \in \mathscr{H}'(I, Z)$ is homotopic, keeping the endpoints fixed, to the map h_0, defined by $h_0(t) = (1 + t, 0)$, for $t \in I$. It then follows from [8, p. 89] that h is isotopic to h_0, keeping the endpoints fixed. Clearly, $p \circ H_t(h) \circ p^{-1}$ is always a homeomorphism of A^2 onto itself, and the homotopy K_t, defined by $K_t(h) = p \circ H_t(h) \circ p^{-1} \circ h$, gives a strong deformation retraction of $\mathscr{H}(I, Z)$ onto $\mathscr{H}'(I, Z)$.

Let D_2 denote the closure of Z, which is just A^2 minus the open disk with radius 1/4 and center $(-3/2, 0)$. Let $\mathscr{H}_0(D_2)$ be the identity component of the space of homeomorphisms of D_2 onto itself, keeping the boundary curves pointwise fixed. For each $h \in \mathscr{H}'(I, Z)$, we are going to define a canonical homeomorphism $k(h) \in \mathscr{H}_0(D_2)$, where "canonical" means that $k(h)$ is uniquely determined by h and varies continuously with h. We also want $k(h)$ to map $(1 + t, 0)$ to $h(t)$, for $t \in I$.

First, we define a homeomorphism $k'(h)$ of D_2' onto itself, where D_2' is A' minus the collection of open disks above the open disk with radius 1/4 and center $(-3/2, 0)$. The space D_2' is a covering space of D_2, and we use the same symbol p

for the restriction of the previous covering projection. We shall define $k'(h)$ on the closed annular region consisting of points (u_1, u_2) with u_1 between 1 and 2, u_2 between 0 and 2π, and the distance from (u_1, u_2) to $(3/2, \pi) \geqq 1/4$. We then use translates of this definition on the rest of D_2'. The homeomorphism $k'(h)$ will be uniquely determined by the following four boundary conditions:

(1) $k'(h)$ is the identity on the line segments from $(1, 0)$ to $(1, 2\pi)$ and from $(2, 0)$ to $(2, 2\pi)$,

(2) $k'(h)(1+t, 0) = h'(t)$ and $k'(h)(1+t, 2\pi) = h''(t)$, for $t \in I$,

(3) $k'(h)$ is the identity on the circle with radius $1/4$ and center $(3/2, \pi)$, and

(4) the angle change is zero, as determined by the image of the segment from $(1, \pi)$ to $(5/4, \pi)$ and by using the points $(3/2, \pi)$ as origin for polar coordinates.

We define $k(h)$ to be the homeomorphism $p \circ k'(h) \circ p^{-1}$. In §1, we stated that $k'(h)$ has the continuity property ($k'(h)$ varies continuously with h), and it is clear that $k(h)$ also has this continuity property.

Since any map $h \in \mathscr{H}'(I, Z)$ is isotopic, keeping the endpoints fixed, to h_0, where $h_0(t) = (1+t, 0)$, for $t \in I$, we see that k will induce a path of homeomorphisms of D_2, keeping the boundary curves fixed, from $k(h)$ to the identity on D_2. Hence $k(h)$ belongs to $\mathscr{H}_0(D_2)$.

It was proved in [15] that $\mathscr{H}_0(D_2)$ is contractible in itself. (For Theorem 3.2(c) we only need the result from [10] that $\mathscr{H}_0(D_2)$ has vanishing homotopy groups.) Using the map k, the contraction of $\mathscr{H}_0(D_2)$ induces a contraction of $\mathscr{H}'(I, Z)$, which is a deformation retract of $\mathscr{H}(I, Z)$. Hence the latter space is contractible. Q.E.D.

Next we prove a result which is perhaps known in some general formulation to experts, but has apparently not appeared in print.

3.4. Lemma. *The space $\mathscr{H}(I, A^2)$ is locally contractible.*

Proof. Let h_1 be any map in $\mathscr{H}(I, A^2)$, and let U be the compact-open neighborhood of h_1 consisting of maps h whose distance in the sup-metric from h_1 is less than ε, for some $\varepsilon > 0$. Corresponding to each $h \in U$, we shall produce a homeomorphism $k(h) \in \mathscr{H}(A^2)$, the space of homeomorphisms of A^2 onto itself, not necessarily fixing the boundary. For any $h \in U$, $k(h)$ will be uniquely determined by the following four conditions:

(1) $k(h)$ is a rotation on C_1 so that $k(h)(h_1(0)) = h(0)$,

(2) $k(h)$ is a rotation on C_2 so that $k(h)(h_1(1)) = h(1)$,

(3) $k(h)(u) = h \circ h_1^{-1}(u)$, for $u \in h_1(I)$, and

(4) $k(h)(u_0) = u_0$, for some fixed u_0 in the interior of A^2 which misses all the sets $h(I)$, for $h \in U$.

We can construct $k(h)$ in a manner similar to the construction of the previous $k(h)$ in Lemma 3.3. As before, $k(h)$ will vary continuously with h, and $k(h_1)$ is the identity map on A^2. Also $k(h) \circ h_1 = h$.

Set U' equal to the set of all $h' \in \mathscr{H}(A^2)$ within a distance ε of the identity map on A^2, where ε is the same number used to define U. The space $\mathscr{H}(A^2)$ is locally contractible [11, p. 524], so let W' be a neighborhood of the identity which is contractible in U'. Let W be a neighborhood of h_1 in U such that $k(W) \subset W'$, and let $W' \circ h_1$ be the set of all $h \in \mathscr{H}(I, A^2)$ such that $h = h' \circ h_1$, for some $h' \in W'$. It is easy to see that $W \subset W' \circ h_1 \subset U' \circ h_1 \subset U$, and thus $W' \circ h_1$ is a neighborhood of h_1 in U which is contractible in U. Q.E.D.

Proof of Theorem 3.2. Using a technique from [7, p. 107], let us first show that $\mathscr{H}(I, A^2)$ is topologically complete. The space $\mathscr{C}(I, A^2)$ of all continuous functions of I into A^2 is complete in the sup-metric, and it has as a closed subspace the space $\mathscr{C}_0(I, A^2)$ of all maps taking 0 into C_1 and 1 into C_2. For each positive integer n, consider the subspace of $\mathscr{C}_0(I, A^2)$ consisting of all functions g such that $\sup\{\text{diam}(g^{-1}(u)) : u \in g(I)\}$, $\sup\{\alpha : g(\alpha) \in C_1\}$, and $1 - \inf\{\alpha : g(\alpha) \in C_2\}$ are all less than $1/n$. These subspaces are open in $\mathscr{C}_0(I, A^2)$ and their intersection is $\mathscr{H}(I, A^2)$, so that the latter, as a G_δ in $\mathscr{C}_0(I, A^2)$, is topologically complete.

A proof very similar to that of Lemma 2.2 will show that $\mathscr{B}$ is open in $I^n \times \mathscr{H}(I, A^2)$, and hence is topologically complete. The rest of the theorem now follows easily from Lemmas 3.3 and 3.4 as in the proof of Lemma 2.3. Q.E.D.

Proof of Theorem 3.1. Please refer to the statement of Theorem 3.2 and the discussion immediately preceding it. Let $g_0 : I \to A^2$ be the map defined by $g_0(t) = (-1-t, 0)$, for $t \in I$. Since $\Phi(y) = \Lambda(3/2, 0)$, for $y \in \partial I^n$, we see that $g_0(I) \subset M(\Phi(y))$ or $(y, g_0) \in \mathscr{B}$, when $y \in \partial I^n$. Let $e : \partial I^n \to \mathscr{B}$ be the map $e(y) = (y, g_0)$. Theorem 1.3 gives an extension $e^* : I^n \to \mathscr{B}$. We have arranged things so that, for any $x \in I^n$, $\tau_2 \circ e^*(x)$ is an element of $\mathscr{H}(I, M(\Phi(x)))$, where τ_2 is the projection of $I^n \times \mathscr{H}(I, A^2)$ onto $\mathscr{H}(I, A^2)$. Thus, $\tau_2 \circ e^*(x)$ gives the desired canonical arc from C_1 to C_2 in $A^2 \backslash \text{im}(\Phi(x))$. Notice also that $\Phi(x)(3/2, 0)$ gives a canonical point in the interior of A^2 which misses the image of $\tau_2 \circ e^*(x)$.

By the same construction that was used in the proofs of Lemmas 3.3 and 3.4 (using the universal covering space of A^2), we get a canonical homeomorphism $k(x)$ of A^2 onto itself, for each $x \in I^n$. The homeomorphism $k(x)$ will be uniquely determined by the following four conditions:

(1) $k(x)$ is a rotation on C_1 taking $\tau_2 \circ e^*(x)(0)$ to $(-1, 0)$,
(2) $k(x)$ is a rotation on C_2 taking $\tau_2 \circ e^*(x)(1)$ to $(-2, 0)$,
(3) $k(x) \circ (\tau_2 \circ e^*(x)) = g_0$, and
(4) $k(x)(\Phi(x)(3/2, 0)) = (3/2, 0)$.

The above construction yields a map $\Phi' : (I^n, \partial I^n) \to (\mathscr{N}(A^2), \Lambda(3/2, 0))$, defined by $\Phi'(x) = k(x) \circ \Phi(x) \circ k(x)^{-1}$, whose image consists of retractions with images in the complement of the segment from $(-1, 0)$ to $(-2, 0)$. We now cut A^2 along this segment to yield a closed disk with a collection of retractions, and then use Theorem 1.1 to see that these are homotopic to the constant retraction to $(3/2, 0)$. From the proof of Theorem 1.1, it is clear that the retractions of this disk will yield retractions of A^2, after identifying the cut portion. Thus Φ' is homotopic to a constant map with image $\Lambda(3/2, 0)$. This defines a homotopy of Φ to a map Ψ such that

$$\begin{aligned}\Psi(x) &= k(x)^{-1} \circ \Lambda(3/2, 0) \circ k(x) \\ &= k(x)^{-1} \circ \Lambda(3/2, 0) = \Lambda(k(x)^{-1}(3/2, 0)).\end{aligned}$$

Hence $\Psi(x)$ lies in the image of Λ, so that Λ_* is surjective. Q.E.D.

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